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Algebraic and scattering aspects of a $\mathcal{PT}$-symmetric solvable potential

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Abstract

We study a particular solvable potential and analyse the effect of $\mathcal{PT}$ symmetry on its bound state as well as scattering solutions. We determine the transmission and reflection coefficients for the $\mathcal{PT}$-symmetric case and also formulate the problem in terms of an $SU(1, 1)$ potential group, which allows unified treatment of the discrete and the continuous spectra in a natural way. We find that (bound and scattering) states of the $\mathcal{PT}$-symmetric problem supply a basis for the unitary irreducible representations of the $SU(1, 1)$ potential group, and this gives a straightforward group theoretical interpretation of the fact that the (complex) $\mathcal{PT}$-invariant potential has a real energy spectrum.

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Recently a rather intriguing invariance property of one-dimensional quantum mechanical potentials has been identified. Hamiltonians admitting $\mathcal{PT}$ symmetry [1] are invariant under the simultaneous operations of space and time reflection. $\mathcal{PT}$-invariant problems have been studied in quantum field theory [2], but they were also found to be relevant via mathematical analogies to problems in solid state physics [3] and population biology [4]. For one-dimensional potentials of non-relativistic quantum mechanics $\mathcal{PT}$ invariance requires $(V(-x))^* = V(x)$. Potentials having this invariance property are usually complex; nevertheless, their bound-state energy eigenvalues were found to be real. Such potentials have been identified in various approaches, such as semiclassical [1, 5], numerical [6] and perturbative [7] methods, but exact analytical solutions to some problems have also been given [8–15].

Most of the exactly solvable potentials are known to possess some kind of symmetry property, which are formulated in terms of algebraic constructions (see e.g. [16–19]); therefore, the question of how this new symmetry concept, $\mathcal{PT}$ symmetry, is related to the existing ones emerges naturally. Among the various algebraic schemes the potential group approach [16] deserves special attention for several reasons. First, the energy eigenvalues are obtained from the eigenvalues of the Casimir operator of some algebra, and real eigenvalues of the latter operator are specific to particular (i.e. unitary) irreducible representations in the algebraic framework. Second, in the potential group approach bound-state and scattering solutions can be described in a unified way, i.e. in terms of the continuous and the discrete unitary irreducible representations of a non-compact potential group [16]. This also opens the way to the analysis of scattering solutions of $\mathcal{PT}$-symmetric potentials, which, to our knowledge, have not been studied previously, except for some considerations concerning transparent $\mathcal{PT}$-symmetric potentials [20, 21].

For our analysis we select an exactly solvable, but non-trivial potential which (i) is non-singular on the $x \in (-\infty, \infty)$ domain, (ii) has a $\mathcal{PT}$ symmetric version and (iii) has a non-compact potential group associated with it. A potential of this kind is

$$V(x) = (\lambda^2 - s(s+1))\frac{1}{\cosh^2 x} + \lambda(2s+1)\frac{\sinh x}{\cosh^2 x}. \tag{1}$$

Curiously enough, this potential, which is a natural non-symmetric extension of the Pöschl–Teller potential, has been overlooked in the literature, and it was studied first [22, 23] within the context of supersymmetric quantum mechanics [24]. Here we have slightly simplified the notation of [22, 23] in order to avoid heavier formulae later on.

The scattering amplitudes for the potential (1) have been studied in [23]. Here we reproduce the formulae necessary for our purposes later on. Applying a form slightly different from that in [23], the two independent solutions of the Schrödinger equation are

$$F_1(x) = (1+\mathrm{i}y)^{-\frac{s-\mathrm{i}\lambda}{2}}(1-\mathrm{i}y)^{-\frac{s+\mathrm{i}\lambda}{2}} F\left(-s-\mathrm{i}k, -s+\mathrm{i}k, \mathrm{i}\lambda - s + \frac{1}{2}; \frac{1+\mathrm{i}y}{2}\right) \tag{2}$$

and

$$F_2(x) = (1+\mathrm{i}y)^{\frac{s+1-\mathrm{i}\lambda}{2}}(1-\mathrm{i}y)^{-\frac{s+\mathrm{i}\lambda}{2}} F\left(\frac{1}{2} - \mathrm{i}\lambda - \mathrm{i}k, \frac{1}{2} - \mathrm{i}\lambda + \mathrm{i}k, s + \frac{3}{2} - \mathrm{i}\lambda; \frac{1+\mathrm{i}y}{2}\right) \tag{3}$$

with $y(x) = \sinh x$. According to [23] the transmission and reflection amplitudes are

$$T(k, s, \lambda) = \frac{\Gamma(-s-\mathrm{i}k)\Gamma(s+1-\mathrm{i}k)\Gamma(\frac{1}{2}+\mathrm{i}\lambda-\mathrm{i}k)\Gamma(\frac{1}{2}-\mathrm{i}\lambda-\mathrm{i}k)}{\Gamma(-\mathrm{i}k)\Gamma(1-\mathrm{i}k)\Gamma^2(\frac{1}{2}-\mathrm{i}k)} \tag{4}$$

$$R(k, s, \lambda) = T(k, s, \lambda)\left(\frac{\cos(\pi s)\sinh(\pi\lambda)}{\cosh(\pi k)} + \mathrm{i}\frac{\sin(\pi s)\cosh(\pi\lambda)}{\sinh(\pi k)}\right). \tag{5}$$

(We have corrected a sign error in the denominator of (4) in equation (18*a*) of [23].) From the analysis of the poles of the T coefficient, it turns out that potential (1), for $s > 0$, has bound-state energy eigenvalues at

$$E_n = -(s-n)^2 \tag{6}$$

corresponding to $-s - \mathrm{i}k = -n$, where n is a positive integer. In this case (2) turns into a polynomial function of $y(x) = \sinh(x)$, which becomes normalizable only for $s > n$. This condition gives rise to a finite number of bound states, as expected for a short-range potential. Note that the energy eigenvalues depend only on the parameter s, and are free from λ.

Potential (1) was also found in a systematic search for shape-invariant potentials [25], while in [19] an $su(1, 1)$ potential algebra was assigned to it, with generators

$$J_\pm = \mathrm{e}^{\pm\mathrm{i}\phi}\left(\pm\frac{\partial}{\partial x} - \tanh x\left(J_z \pm \frac{1}{2}\right) - \frac{\lambda}{\cosh x}\right) \qquad J_z = -\mathrm{i}\frac{\partial}{\partial\phi}. \tag{7}$$

The basis functions are then written as

$$\langle x|jm\rangle = \mathrm{e}^{\mathrm{i}m\phi}\psi_{jm}(x). \tag{8}$$

Besides carrying the extra phase factor, these wavefunctions are essentially the same as those in (2). For bound states, the $m = s + \frac{1}{2}$ and $j = n - m$ substitutions have to be made. A similar realization of the $SU(1,1)$ generators was also proposed later in [26], with λ being an arbitrary complex parameter leading to non-unitary irreps.

The generators defined as in (7) satisfy the $su(1,1)$ algebra

$$[J_z, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = -2J_z. \tag{9}$$

(We follow the usual notation and use small letters to refer to algebras.) The basis states (8) are eigenstates of the J_z operator with eigenvalue m, and of the Casimir operator

$$\begin{aligned} C_2 &= -J_+J_- + J_z^2 - J_z \\ &= -J_-J_+ + J_z^2 + J_z \end{aligned} \tag{10}$$

with eigenvalue $j(j+1)$. The Casimir operator is related to the Hamiltonian as

$$H = -C_2 - \tfrac{1}{4} \tag{11}$$

which also means that the energy eigenvalues are expressed as $E = -(j + \frac{1}{2})^2$. The ladder operators $J_\pm$ take a state with m and n to another state with $m \pm 1$ and $n \pm 1$, while leaving j and the energy invariant, so this $su(1,1)$ algebra acts [19] as a potential algebra [16] and the bound-state eigenfunctions belong to an infinite-dimensional discrete unitary irreducible representation of the $SU(1,1)$ group denoted by D_j^+, where m is bounded from below: $m = -j + n, n = 0, 1, 2, \ldots$. The action of J_+ (J_-) is then connecting solutions in potentials with increasing (decreasing) depth, with also increasing (decreasing) principal quantum number n. In [19] it was also shown that the generators J_+ and J_- essentially have the same action as the supersymmetric shift operators $A^\dagger$ and A, which also connect states of the same energy in potentials having different depths.

While bound states are assigned to discrete unitary irreducible representations in the $SU(1,1)$ potential group approach, scattering solutions are related to the continuous unitary irreducible representation C_j^δ [16]. In this case $j = -\frac{1}{2} + \mathrm{i}k$ holds, so the Casimir invariant has negative eigenvalues $\langle C_2 \rangle = j(j+1) = -\frac{1}{4} - k^2$; consequently, due to (11) the corresponding energy eigenvalues are positive. The label m which sets the potential strength (via $m = s + \frac{1}{2}$) can now take integer ($\delta = 0$) or half-integer ($\delta = \frac{1}{2}$) values.

Besides the bound and scattering solutions belonging to the unitary irreducible representations of $SU(1,1)$, there are also resonance solutions appearing as poles of the transmission amplitude (4) with complex wavenumber $k = \pm\lambda - \mathrm{i}(n + \frac{1}{2})$. These belong to *non-unitary* irreducible representations of the $SU(1,1)$ potential group [27].

One-dimensional potentials incorporating $\mathcal{PT}$ symmetry have to satisfy the relation $\tilde{V}(x) \equiv (V(-x))^* = V(x)$. The $\mathcal{P}$ space reflection operation carries x into $-x$, while $\mathcal{T}$, the time reflection corresponds to complex conjugation. For some ordinary potentials the substitution $x \to x + \mathrm{i}\epsilon$ [11, 12] together with the replacement of the originally real coupling constants of the odd terms with imaginary ones [11] secures $\mathcal{PT}$ invariance. This is the case also with potential (1) [15]. This results in a non-Hermitian (complex) potential

$$\tilde{V}(x) = (-\tilde{\lambda}^2 - s(s+1))\frac{1}{\cosh^2(x + \mathrm{i}\epsilon)} + \mathrm{i}\tilde{\lambda}(2s+1)\frac{\sinh(x + \mathrm{i}\epsilon)}{\cosh^2(x + \mathrm{i}\epsilon)}. \tag{12}$$

A special case of this potential (with $\epsilon = 0$) was mentioned in [21]. We note that the imaginary coordinate shift $\mathrm{i}\epsilon$ can be interpreted in several ways. First, one can argue that the problem is shifted to a domain of the whole complex x plane and in this sense it becomes similar to other $\mathcal{PT}$-invariant potential problems, for which the allowed x values are restricted to some wedges in the x plane [1, 5, 10]. Second, however, one can also interpret it as a conventional complex potential defined on the x axis, because the potential terms can be easily rewritten in a form which contains ϵ as a potential parameter (typically appearing in the argument of hyperbolic and trigonometric functions), rather than an imaginary coordinate shift. See, for example,

$$\cosh(x + \mathrm{i}\epsilon) = \cos\epsilon \cosh x + \mathrm{i} \sin\epsilon \sinh x \tag{13}$$

and a similar expression for $\sinh(x + \mathrm{i}\epsilon)$. In the former case special attention has to be paid to the interpretation of the wavefunctions, since some fundamental quantities, such as the norm of a wavefunction, might require non-standard definitions [28] when the potentials are defined on the whole complex x plane. In particular, in [28], the analytic function $\psi^2(x)$ replaces the standard $|\psi(x)|^2$ in the definition of the norm: this suggests that also the quantum flux in the continuity equation should be redefined in the complex x plane [5]. In the latter case, however, no such complications occur, and one can follow the usual methods of dealing with complex potentials of a real argument. In what follows we consider this latter option and interpret our $\mathcal{PT}$-invariant potential as a problem defined on the x axis.

Let us now study the implications of $\mathcal{PT}$ symmetry for the scattering solutions of this problem. If we consider only the $\lambda \to \mathrm{i}\tilde{\lambda}$ substitution, then the forms of the transmission and reflection amplitudes (4) and (5) remain unchanged. If we also apply the transformation $x \to \tilde{x} = x + \mathrm{i}\epsilon$, then $T(k, s, \lambda)$ remains unchanged, while $R(k, s, \lambda)$ in (5) is modified by a multiplicative factor $\exp(2\epsilon k)$. This clearly shows that the $|T|^2 + |R|^2 = 1$ relation breaks down in the $\mathcal{PT}$ symmetric case, which is not astonishing if we recall that we have complex potentials in this case, in which the flux is not conserved. We note that although the extra $\exp(2\epsilon k)$ factor increases the modulus of the reflection amplitude (5) if $\epsilon k > 0$, it remains finite as long as $\epsilon < \pi/2$. Since $\cosh(x + \mathrm{i}\frac{\pi}{2}) = \mathrm{i}\sinh(x)$ (see also equation (13)), in this limit (12) becomes [15] a singular (generalized Pöschl–Teller) potential, and equations (4) and (5) do not apply.

Similar considerations hold in the case of negative ϵ, which would lead to a decrease of the reflection coefficient.

The $\lambda = \mathrm{i}\tilde{\lambda}$ substitution has no impact on the bound states, the energies of which are obtained from the poles of $T(k, s, \lambda)$ on the positive imaginary k axis with $k = \mathrm{i}(s - n)$. Note, however, that the resonance states present in the Hermitian case at $k = \pm\lambda - \mathrm{i}(n + \frac{1}{2})$ become states with *real* energy, when $\tilde{\lambda} - \frac{1}{2} - n > 0$ holds. This duplication of states seems interesting, especially in light of the fact that in the Hermitian case the energy eigenvalues did not depend on λ at all. However, we can note that the potential (1) has a symmetry with respect to the replacement $s \leftrightarrow \mathrm{i}\lambda - \frac{1}{2}$. This transformation changes the two independent solutions (2) and (3) into each other, and replaces the bound-state solutions with the resonance-state solutions described above.

In the $\mathcal{PT}$-symmetric case, however, the two sets of solutions both have negative real energies. This might be due to the fact that the real part of the potential deepens when λ has imaginary values. In any case, it is remarkable that even those energy eigenvalues which are complex in the Hermitian case become real when $\mathcal{PT}$ symmetry is imposed on the system. This suggests that the reality of the energy eigenvalues is really very tightly related to $\mathcal{PT}$ symmetry. One interesting aspect of the duplication of the bound-state energy spectrum is that for $s = -\tilde{\lambda} - \frac{1}{2}$ the energy levels coincide pairwise. In fact, the two types of solution become *identical* in this case (see (2) and (3)), which means that the general theorem which forbids the degeneracy of energy levels in one-dimensional potentials is not violated. We note that similar level crossings are also present in some other $\mathcal{PT}$-symmetric potentials [13, 14].

Now let us analyse what happens with the $su(1, 1)$ algebra in the $\mathcal{PT}$ symmetric case.

With the $x \to \tilde{x} \equiv x + \mathrm{i}\epsilon$ and $\lambda \to \mathrm{i}\tilde{\lambda}$ operations, $J_\pm$ become

$$\tilde{J}_\pm = \mathrm{e}^{\pm\mathrm{i}\phi}\left(\pm\frac{\partial}{\partial x} - \tanh(x+\mathrm{i}\epsilon)\left(J_z \pm \frac{1}{2}\right) - \frac{\mathrm{i}\tilde{\lambda}}{\cosh(x+\mathrm{i}\epsilon)}\right) \tag{14}$$

while $J_z = \tilde{J}_z$ remains unchanged. (In order to simplify the notation we made use of the fact that derivation with respect to $\tilde{x}$ has the same effect as derivation with respect to x.) Simple calculations show that

$$\mathcal{PT}\tilde{J}_\pm(\mathcal{PT})^{-1} = \tilde{J}_\mp \qquad \mathcal{PT}\tilde{J}_z(\mathcal{PT})^{-1} = -\tilde{J}_z \tag{15}$$

while the bound-state solutions remain unchanged, and only the phase factor $\mathrm{e}^{\mathrm{i}m\phi}$ changes to $\mathrm{e}^{-\mathrm{i}m\phi}$ in (8). Note that the $\mathcal{PT}$ operation has the same effect on the two ladder operators as Hermitian conjugation in the usual (Hermitian) case. The invariance of the Hamiltonian is reached by the invariance of the Casimir invariant in both cases. However, while in the Hermitian case the two lines of (10) are transformed into themselves by Hermitian conjugation, the invariance of C_2 under the $\mathcal{PT}$ operation is obtained by transforming the two lines of (10) into each other.

One finds that the states of the $\mathcal{PT}$ invariant problem supply a basis for the unitary irreducible representations of the $SU(1,1)$ potential group. Determining, for example, the eigenvalues of $\tilde{J}_+\tilde{J}_-$ and $\tilde{J}_-\tilde{J}_+$ by directly acting with the generators on the basis functions, we find that they are real and negative, as expected for unitary representations of the present $SU(1,1)$ group [29]. The real energy eigenvalues arise as the consequence of the $\mathcal{PT}$ invariance of the problem. It is remarkable that even those states which, in the Hermitian case, appear as resonances (and are assigned to non-unitary irreducible representations) develop into real-energy bound states due to $\mathcal{PT}$ invariance. This has interesting group theoretical implications: it can be shown that a second $su(1,1) \simeq so(2,1)$ algebra can also be associated with the present problem, now with a basis provided by the solutions of type (3). Changing the originally real λ to imaginary values $\mathrm{i}\tilde{\lambda}$ turns the corresponding non-unitary irreps of this latter $SU(1,1)$ group into unitary ones, thus converting the resonances to bound states. The direct sum of the two $so(2,1)$ algebras is isomorphic to an $so(2,2)$ algebra, which is known to be related to the Natanzon potential class [17, 18], to which potential (1) also belongs.

In summary, we have studied the effect of $\mathcal{PT}$ symmetry for an exactly solvable quantum mechanical potential which has a non-compact $SU(1,1)$ potential group associated with it. Besides the bound-state solutions of this complex potential, we have examined also the effect of $\mathcal{PT}$ symmetry on the scattering solutions, a task which practically has not been investigated previously. This is probably because many $\mathcal{PT}$-symmetric problems studied earlier are confining potentials. Defining the problem as a complex potential restricted to the x axis, the transmission amplitude turns out to have the same functional form as in the Hermitian case, while the reflection amplitude picks up an extra factor, violating the unitarity condition $|R|^2 + |T|^2 = 1$.

The energy eigenvalues are found to be real, as is the case with $\mathcal{PT}$-invariant potentials in general. However, a new development is that even those states which appear as resonances and possess complex energy eigenvalues in the Hermitian case become bound states with negative real energy eigenvalues in the $\mathcal{PT}$-symmetric version of the problem. This shows that the link between $\mathcal{PT}$ symmetry and the reality of the energy eigenvalues is even stronger than known previously.

We have also analysed how the algebraic construction changes when the $\mathcal{PT}$-invariance requirement is imposed on the potential. We have found, in analogy with the Hermitian version of the problem, that the states of the $\mathcal{PT}$-symmetric potential supply a basis for the unitary irreducible representations of the $SU(1,1)$ potential group, in this case too. This gives a natural explanation for the real energy eigenvalues.

Results in agreement with our findings were independently obtained in [30], where bound states, but not scattering states, of several $\mathcal{PT}$-symmetric potentials were analysed in terms of a complexified $so(2,1)$ algebra.

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